University of California, Berkeley
Physics 110B, Spring 2004 (*Strovink*)

Exercises 53-59 are devoted to Jones matrices and to the Stokes vectors used to represent partially polarized light.

53.
(a.)
A set of N ideal linear polarizers $\mathrm{L}_1 \ldots \mathrm{L}_N$ is arranged so that $\hat{x}$ polarized light passes through them in ascending order. The transmission axis of polarizer n is oriented along $(\hat{x}\cos\phi_n + \hat{y}\sin\phi_n)$, where $\phi_n = \frac{\pi n}{2N}$. In the limit $N \to \infty$, deduce the Jones matrix for this set.
(b.)
Consider a *twisted nematic cell*, as found in an LCD display. It functions as a *rotator* (Pedrotti×2 Eq. (14-21)). Show that if the rotator parameter $\beta = \frac{\pi}{2}$, the twisted cell will have the same effect on $\hat{x}$ polarized light as does the set of polarizers described in (a.).
(c.)
Do the devices in (a.) and (b.) also have equivalent effect on $\hat{y}$ polarized light? Explain.

54.
Apart from an experimentally irrelevant overall phase, an ideal wave plate of thickness D with phase retardation difference

$$\delta \equiv (n_x - n_y)\frac{\omega D}{c} \ ,$$

having its slow axis along $\hat{x}$, is represented by the Jones matrix

$$M_{\mathrm{W}}(\phi = 0) = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix} \ .$$

If instead the wave plate has its slow axis along $(\hat{x}\cos\phi + \hat{y}\sin\phi)$, show that it is represented by the general Jones matrix

$$M_{\mathrm{W}}(\phi) = \begin{pmatrix} \cos\frac{\delta}{2} + i\sin\frac{\delta}{2}\cos 2\phi & i\sin\frac{\delta}{2}\sin 2\phi \\ i\sin\frac{\delta}{2}\sin 2\phi & \cos\frac{\delta}{2} - i\sin\frac{\delta}{2}\cos 2\phi \end{pmatrix}$$

Note that $\delta = \frac{\pi}{2}$ for a quarter-wave plate (QWP) and $\delta = \pi$ for a half-wave plate (HWP), which is equivalent to two QWPs. Note also that, like the general Jones matrix $M_{\mathrm{L}}(\phi)$ for the ideal linear polarizer (Pedrotti×2 Eq. 14-15), $M_{\mathrm{W}}(\phi)$ is *symmetric* and invariant to the transformation $\phi \to \phi + \pi$. However, unlike $M_{\mathrm{L}}(\phi)$, $M_{\mathrm{W}}(\phi)$ is also *unitary* ($M^{-1} = M^\dagger$) with unit determinant, preserving the irradiance.

55.
Use the result of the previous problem to do Pedrotti×2 Problem 14-11. To get their result you must assume, as they do [Eqs. (14-17)-(14-20)], that the wave plate's slow axis lies along either the x or y axis.

56.
(a.)
Do Pedrotti×2 Problem 14-17. Does their Jones matrix really convert *any* state of incident polarization to a *finite* irradiance of RH polarized light? Explain.
(b.)
Devise a combination of ideal wave plate(s) and polarizer(s) that, within a multiplicative constant, yields the Jones matrix of part (a.). Supply the absolute magnitude of this constant. Congratulations! You have invented an ideal *homogeneous right-hand circular polarizer.*
(c.)
Show that the result of part (b.) functions also as a *right-hand circular analyzer*, *i.e.* it fully transmits RH circularly polarized light and fully absorbs LH circularly polarized light.

57. Stokes vectors #1.
Using the standard definition of the complex electric field $\vec{E}_0$,

$$\vec{E}(z,t) = \mathrm{Re}\big(\vec{E}_0 \exp\left(i(\tilde{k}z - \omega t)\right)\big) \ ,$$

consider the case in which the phase difference between its x and y components

$$\epsilon(t) = \arg E_{0x} - \arg E_{0y}$$

is not necessarily fixed, as would be the case for fully polarized light, but rather is allowed to vary with time – slowly with respect to ω^{-1}, but rapidly with respect to experimenters' ability to measure it. The *Stokes vector* $\mathcal{S}$ is defined by the real elements

$$\mathcal{S} \equiv \begin{pmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix} \equiv \frac{\mathrm{Re}\,\tilde{k}}{\mu\omega} \begin{pmatrix} |E_{0x}|^2 + |E_{0y}|^2 \\ |E_{0x}|^2 - |E_{0y}|^2 \\ \langle 2\,\mathrm{Re}(E_{0x}E_{0y}^*)\rangle \\ \langle 2\,\mathrm{Im}(E_{0x}E_{0y}^*)\rangle \end{pmatrix} ,$$

where $\langle\rangle$ denotes the time average.

(a.)

Show that

$$\mathcal{S} = \frac{\mathrm{Re}\,\tilde{k}}{\mu\omega} \begin{pmatrix} |E_{0x}|^2 + |E_{0y}|^2 \\ |E_{0x}|^2 - |E_{0y}|^2 \\ \langle 2|E_{0x}||E_{0y}|\cos\epsilon\rangle \\ \langle 2|E_{0x}||E_{0y}|\sin\epsilon\rangle \end{pmatrix} .$$

(b.)

The *normalized Stokes vector* $\bar{\mathcal{S}}$ is defined to be the usual Stokes vector divided by $\mathcal{S}_0$, so that its topmost element is unity. Consider a fully polarized beam in an arbitrary state of polarization described by the general Jones vector

$$J = \frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} .$$

Show that the normalized Stokes vector for this beam is

$$\bar{\mathcal{S}} = \frac{1}{|\alpha|^2 + |\beta|^2} \begin{pmatrix} |\alpha|^2 + |\beta|^2 \\ |\alpha|^2 - |\beta|^2 \\ 2\,\mathrm{Re}(\alpha\beta^*) \\ 2\,\mathrm{Im}(\alpha\beta^*) \end{pmatrix} .$$

(c.)

Using the result of (b.) and your knowledge of Jones vectors, show that fully linearly polarized beams in the $\hat{x}$, $\hat{y}$, $\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$, and $\frac{1}{\sqrt{2}}(\hat{x}-\hat{y})$ directions are described, respectively, by the normalized Stokes vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} ,$$

and that fully circularly RH and LH polarized beams are described, respectively, by the normalized Stokes vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} .$$

58. Stokes vectors #2.

Please refer to the notation and results of the previous problem.

(a.)

For fully polarized ("p") light (ϵ fixed), show that

$$\mathcal{S}_1^2 + \mathcal{S}_2^2 + \mathcal{S}_3^2 = \mathcal{S}_0^2 .$$

(b.)

Natural ("n") light is completely unpolarized. It has $|E_{0x}| = |E_{0y}|$, but the phases of both E_{0x} and E_{0y} vary randomly with time so that $\langle\cos\epsilon\rangle = \langle\sin\epsilon\rangle = 0$. For natural light, show (conversely to (a.)) that

$$\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = 0 .$$

59. Stokes vectors #3.

Please refer to the notation and results of the previous two problems. Consider four devices: (A) a grey filter passing half the incident irradiance; (B) an $\hat{x}$ polarizer; (C) an $\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$ polarizer; (D) a RH circular analyzer. After passing through (only) device X, the beam has irradiance I_{X}. It can be shown that

$$\mathcal{S} = 2 \begin{pmatrix} I_{\mathrm{A}} \\ I_{\mathrm{B}} - I_{\mathrm{A}} \\ I_{\mathrm{C}} - I_{\mathrm{A}} \\ I_{\mathrm{D}} - I_{\mathrm{A}} \end{pmatrix} .$$

Therefore, a Stokes vector can be completely determined by *measuring only irradiances*. This reveals one extent to which Stokes "vectors" satisfy vector properties. The additive property normally associated with a vector, $\mathcal{S}_{\mathrm{tot}} = \mathcal{S}_A + \mathcal{S}_B$ for two beams A and B, holds only if their *irradiances* rather than their amplitudes add, *i.e.* only if the two beams are *completely mutually incoherent*. This is a total contrast to Jones vectors, which can be defined only for fully polarized beams and can be added only if the two beams are completely mutually coherent.

(a.)

Using the additive property for Stokes vectors in mutually incoherent beams, show that an arbitrary beam

$$\mathcal{S} = \begin{pmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix}$$

is the (necessarily incoherent) superposition of a fully polarized beam p and a natural-light beam n. Show this by specifying the elements of the constituent Stokes vectors $\mathcal{S}_p$ and $\mathcal{S}_n$ in terms of the elements of the overall Stokes vector $\mathcal{S}$.

(b.)

Define the *degree of polarization* V by

$$V \equiv \frac{I_p}{I_p + I_n} \; .$$

For the above arbitrary beam, show that

$$V = \frac{\sqrt{\mathcal{S}_1^2 + \mathcal{S}_2^2 + \mathcal{S}_3^2}}{\mathcal{S}_0} \; .$$

Appendix: Mueller matrices

The *Mueller matrices* manipulate Stokes vectors in the same way that Jones matrices manipulate Jones vectors. For an $\hat{x}$ polarizer and for a $\frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ polarizer, the Mueller matrices are, respectively,

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; .$$

The Mueller matrices for a $\hat{y}$ polarizer and for a $\frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$ polarizer are, respectively,

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; .$$

For a QWP with slow axis along x and for a homogeneous right-hand circular polarizer, the Mueller matrices are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \; .$$

The Mueller matrices for a QWP with slow axis along y and for a homogeneous left-hand circular polarizer are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \; .$$